Large Eddy Numerical Simulations of Turbulent Flows

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Large eddy simulations are a numerical technique in which large-scale turbulent structures are computed explicitly, and the small structures are modeled. Arguments for believing this method to be superior to more conventional approaches are given, the basis of the method is given, and some typical results displayed. The results show that the method does have enormous promise, but much further development is required.

I. Rationale

TURBULENCE originates in instabilities of laminar flow, the precise nature of which depend on the geometry of the flow. Generally, the instabilities produce wavelike structures which can absorb energy from the mean flow. As they grow, nonlinear effects cause energy transfers to other modes, and, eventually, the noisy pattern that is generally regarded as "turbulence" results.

Fully developed turbulence always reflects its origins to some degree. Since the maintenance of turbulence requires continuous nourishment, turbulent structures must be capable of absorbing energy from the mean flow. Although the mean flow is changed by the presence of turbulence, the energy absorbing structures bear some resemblance to those from which the turbulence originated.

Thus, in turbulent flows, the large structures are the ones which absorb energy from the mean flow. They tend to be highly anisotropic, vortical in nature, and quite variable from flow to flow, and they are responsible for most of the property transport in turbulent flows. Furthermore, since the production mechanism is largely the stretching of vortices, which is a process requiring three dimensions, we would argue that all true turbulent flows are three dimensional. Through nonlinear interactions, they transfer some of their energy to smaller-scale structures, and the major function of the small structures is to dissipate the energy provided by the larger ones. In contrast, it is well known that the parameters of the small scales are determined almost entirely by the amount of energy they are required to dissipate. For this reason, they are much more universal than the large structures, and they are therefore nearly the same in all flows and nearly isotropic.

Over the past 10 to 20 years, a large body of experimental evidence for the picture just presented has been accumulating. The total picture is, in fact, much more complicated than that described in the foregoing, and it is likely to be some time before many of the missing elements are filled in.

Almost all of the approaches to the prediction of turbulent flow are based on Reynolds' idea of averaging the Navier-Stokes equations over an ensemble of identical flows or some equivalent (time or span averaging) to obtain an equation for the mean velocity. The equations are not closed because of the nonlinear terms in the Navier-Stokes equations, so approximations are needed, i.e., models must be introduced. Since many of the difficulties that occur in this approach have analogs in what we will be doing in this paper, no direct discussion of them will be given here. Rather, the interested reader is referred to the recent review of these methods by Reynolds.[1]

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The major deficiency of this approach is that one is averaging over all of the turbulence simultaneously, so it is necessary to model all of the structures. Since the structures are so different in various flows, it would be surprising if a universal model of turbulence could be found. Our interpretation of the record of achievement of this approach validates this conclusion.

Any improvement in the effectiveness of predictive methods will come at increased cost. A clue to the direction of improved methods can be obtained by looking at the recent trends in experimental work. Conditional sampling techniques, in which one averages data over time periods in which only one type of event occurs, have been more successful in sorting out phenomena than full averaging methods. The approach used in this paper may be regarded as a crude form of conditional sampling and the experimental experience may prove a valuable guide in further improving predictive methods.

The basic idea on which we will rely is that the large eddies cannot and should not be modeled, but the small eddies might be modeled successfully. The method is initiated by the introduction of an averaging procedure which separates the large- and small-scale structures. The large scales will be computed explicitly, whereas the small scales necessarily are modeled. This approach requires considerably more computation than other methods, but it is hoped that it is more universal. For obvious reasons, we have called it *large eddy simulation.*

Any turbulent flow contains structures ("eddies") with a wide range of scales and no clear distinction can be made between "large" and "small" eddies. Obviously, the largest eddies behave very much in the manner attributed to the "large eddies" and the smallest behave like "small eddies." Between them is a continuous spectrum of eddies sharing the properties of both with a gradual transition from one kind of behavior to the other. Our ability to segregate eddies into two classes requires only that the structures which we call "small eddies" behave in the manner ascribed to them. Some authors have claimed that the existence of an "inertial subrange" is required for the method to work, but our results seem to show that this is an unnecessarily strict condition.

The method is more expensive computationally than other approaches in common use. However, our experience has shown that, at least for simple flows, good results can be obtained with a surprisingly small number of mesh points and, consequently, the method is more expensive than more conventional Reynolds stress modeling by only a factor of two or three in many cases. With continuing decreases in the cost of computation, we believe that this method has an excellent chance of becoming a computational tool for benchmarking and final design optimization.

In the next section, we introduce the basic approach and review the versions of the method developed by various authors. In Sec. III we will review the numerical methods that

have been used in this approach. In Sec. IV we will cover the results that have been obtained to date, and in the final section we will give some estimates of what can be expected in the future.

II. Large Eddy Simulation Approach

Historically, large eddy simulation arose from the needs of the meteorologists who attempt to predict the global weather pattern. Since the smallest turbulence scales in the atmosphere are of the order of centimeters, there is clearly no hope of ever computing all of the details of the atmosphere. The earliest work of this type was done soon after the first computer of sufficient size to handle a two-dimensional global calculation became available. Several groups have been active in this area, but Smagorinsky[2] generally is credited with the initial development of much of the approach. Since 1963, a number of global atmosphere models have been developed. The method also has been applied by Deardorff[3-5] to the atmospheric boundary layer in order to obtain boundary conditions for global models. Although much of the development of this method took place in meteorology, we will give it only brief mention and concentrate on engineering applications. It also should be noted that this paper is of a semi-review nature and much of the material is not, in fact, original with the author.

The first application of the method to problems of engineering interest also was made by Deardorff,[6] who treated the channel flow with reasonable success. This same problem has been treated with a number of improvements by Schumann.[7]

Orszag and his colleagues have made a number of contributions related to this area, especially in the development of numerical methods. They have generally considered flows with low Reynolds numbers based on turbulence quantities and have been interested particularly in studying turbulence theories.

Our own work at Stanford has been directed toward the understanding of the method, the interaction between physics and numerical methods, and the testing of models. The results obtained are an interesting complement to experimental results and have produced some insights into the nature of turbulence. This work is only now in press, but we will review it along with that of the workers mentioned earlier.

All of the work done to date has dealt with incompressible flow; for the meteorological application, compressibility has been handled via the Boussinesq approximation. Thus, we use as our starting point the incompressible Navier-Stokes equations

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}u_iu_j=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\nabla^2u_i \tag{1}$$

together with the continuity equation

$$\partial u_i/\partial x_i=0 \tag{2}$$

where the usual summation convention applies. In this paper, attention is restricted to the three-dimensional case.

At this point, each author has adopted a slightly different approach, but they can be treated within a single conceptual framework as shown by Leonard.[8] The idea is to separate the large and small eddies mathematically, and this can be achieved by filtering, i.e., we define the large scale field by

$$\bar{f}(r)=\int G(r,r')f(r')\,\mathrm{d}V \tag{3}$$

where $G(r,r')$ is a filter function with a characteristic length Δ. The differences among the various approaches can be regarded as differences in the filter chosen.

Deardorff and most of the early workers have used a "box" filter

$$G(r,r')=\begin{cases}1 & |x_i-x_i'|<\Delta/2\\ 0 & \text{otherwise}\end{cases} \tag{4a}$$

which makes $\bar{f}$ a running mean average. Orszag and his colleagues work in Fourier space using a sharp cutoff at some particular wave number. This is equivalent to using

$$G(r,r')=\prod_{i=1}^{3}\frac{\sin(x_i-x_i')/\Delta}{(x_i-x_i')/\Delta} \tag{4b}$$

Schumann[7] uses a method in which he integrates over a small control volume analytically. This method bears some similarity to Deardorff's approach but introduces surface averages which require considerable extra treatment. In our work, we have favored a Gaussian filter mainly for its mathematical properties

$$G(r,r')=(6/\pi\Delta^2)^{3/2}\exp(-6(r-r')^2/\Delta^2) \tag{5}$$

Other choices are possible and, in particular, we point out that the filter need be neither isotropic nor homogeneous, and there are many flows in which neither of these properties is desirable.

Filtering Eqs. (1) and (2) produces

$$\frac{\partial\bar{u}_i}{\partial t}+\frac{\partial}{\partial x_j}\overline{u_iu_j}=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_i}+\nu\nabla^2\bar{u}_i \tag{6}$$

$$(\partial\bar{u}_i/\partial x_i)=0 \tag{7}$$

and we now are faced with the archetypical problem of turbulence computation – the occurrence of an average of a product $\overline{u_iu_j}$. The approach to handling this term that has been taken universally is the direct analog of Reynolds' original treatment for the time-averaged equations, i.e., one defines a subgrid scale velocity u_i' by

$$u_i=\bar{u}_i+u_i'$$

and one then has

$$\overline{u_iu_j}=\bar{u}_i\bar{u}_j+\overline{u_i'\bar{u}_j}+\overline{\bar{u}_iu_j'}+\overline{u_i'u_j'} \tag{9}$$

The last three terms contain the subgrid scale velocity and therefore must be treated by modeling.

For the first term in Eq. (9), it is important to note that for the kind of averaging defined by Eq. (3), $\bar{f}\neq\bar{\bar{f}}$, i.e., a second averaging does not reproduce the result of the first averaging. Consequently, $\overline{\bar{u}_i\bar{u}_j}\neq\bar{u}_i\bar{u}_j$. Now, $\overline{\bar{u}_i\bar{u}_j}$ depends only on the large-scale velocities, $\bar{u}_i$, and can be calculated explicitly. This approach is especially compatible with the use of Fourier transform methods, and we recently have adopted it. Previously, Kwak et al.[9] used a method suggested by Leonard in which a Taylor series is used to obtain the approximation

$$\overline{\bar{u}_i\bar{u}_j}\simeq\bar{u}_i\bar{u}_j+\frac{\Delta^2}{24}\nabla^2\bar{u}_i\bar{u}_j \tag{10}$$

Most authors use only the first term, but we have found that the second term (which we call the Leonard term) is important in producing the correct spectral distribution. It can be included explicitly (as Kwak et al.[9] have done), or it can be noted that the truncation error in many difference schemes is similar to the Leonard term, and one can rely on the truncation error to handle the problem. The staggered mesh scheme used by Deardorff and Schumann does this quite well, perhaps partially explaining their success, but not all schemes do equally well (see Shaanan et al.[10]).

An approximate evaluation of the second and third terms can be made using arguments similar to those in the preceding (Clark et al.[11]).

$$\overline{\bar{u}_iu_j'}\simeq\frac{\Delta^2}{24}\bar{u}_i\nabla^2\bar{u}_j \tag{11}$$

and this (and the term with i,j reversed) may be combined coveniently with the Leonard term to give

$$\overline{\bar{u}_i\bar{u}_j}+\overline{\bar{u}_iu_j'}+\overline{u_i'\bar{u}_j}=\bar{u}_i\bar{u}_j+\frac{\Delta^2}{12}\frac{\partial\bar{u}_i}{\partial x_\ell}\frac{\partial\bar{u}_j}{\partial x_\ell} \tag{12}$$

which also has the considerable advantage of not increasing the order of the differential equation and thus poses no difficulty with boundary conditions.

The final term of Eq. (9) must simply be modeled. One can argue in a number of ways in modeling this term, but in the long run, the question can probably be settled only by direct testing (see later). The effect of this term is very much the same as that of the Reynolds stresses in the time-averaged equations; it represents the effect of subgrid scale eddies on the large-scale eddies. Since the predominant role of the small eddies is to act as acceptors of energy from the large scales and to dissipate it, the transfer of energy from large to small scales is largely a one-way process and appears to the large eddies as a dissipative effect. Thus, as a first guess, one is tempted to use an eddy viscosity model:

$$\overline{u_i'u_j'}=\nu_T\frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)=\nu_T\bar{S}_{ij} \tag{13}$$

Relationships for the eddy viscosity ν_T can be obtained by analogy with the commonly used eddy viscosity models for the Reynolds stress. For a homogeneous flow, one probably can use a constant, but for more complex flows, a more sophisticated model is probably desirable. Since ν_T is dimensionally L^2T^{-1}, and a natural length scale for the small eddies is provided by the width Δ of the filter, we need find only a time scale and the natural choice is a velocity derivative. This leads naturally to two possibilities:

$$\nu_T=(c\Delta)^2(\bar{S}_{ij}\bar{S}_{ij})^{1/2} \tag{14a}$$

$$\nu_T=(c\Delta)^2(\bar{\omega}_i\bar{\omega}_i)^{1/2} \tag{14b}$$

The first of these is due to Smagorinsky.[2] The second, which we have called the vorticity model, has the useful property of being zero in irrotational regions but is less physical in its origins. We have used both models and, in cases tested to date, have detected no significant difference between the models.

To proceed to a higher level of modeling, one can derive equations for the subgrid scale Reynolds stresses $\overline{u_i'u_j'}$ or any of the other quantities appearing in the foregoing equations. This approach was found necessary by Deardorff[3] because, for the atmospheric boundary layer, the filter width Δ was required to be so large (see the following section for considerations determining Δ) that a very large fraction of the total transport of important quantities was carried by the subgrid scale turbulence, and the eddy diffusion model was found to be insufficient.

In this approach, the equations for $\overline{u_i'u_j'}$ contain higher-order correlations (for example $\overline{u_i'u_j'u_k'}$ and $\overline{p'\partial u_i'/\partial x_j}$) which must be modeled by some kind of closure assumption. Since these equations contain terms similar to those in the equations for ensemble-averaged Reynolds stresses, they can be modeled in an analogous manner. Without going into the details, we note that Deardorff did find improvement, although the agreement with experiment was not as good as he would have desired.

In ensemble-average calculations, closure schemes intermediate between eddy viscosity and full Reynolds stress modeling have been used and, of course, methods similar to those can be used in subgrid scale modeling. Schumann[12] has given a version of a model that would be called turbulence kinetic energy or one equation model in ensemble-averaged calculations; he noted that it is difficult to conclude that any improvement was obtained.

To close this section, we note that many of the points presented are under detailed investigation and represent the author's current opinion and should not be regarded as agreed upon established results.

III. Numerical Methods

In computations of the type discussed in this paper, the interaction between the physics of the problem and the numerical methods used is more severe than in almost any other type of flow computation. The reasons for this are at least threefold and have their origins in the nature of turbulence.

First, turbulence contains important structures over the full range of wavenumbers in all parts of the flow. An immediate consequence is that discussion of the accuracy of numerical methods in terms of the order of truncation error may not be appropriate. The order of the truncation error determines how accurately a numerical method differentiates long waves. Most methods produce values of the first derivative which are zero at $k=\pi/h$, the largest wavenumber that can be represented on a grid. Thus, a better description of a method is given by the accuracy with which sinusoidal functions are differentiated over the full range of wavenumbers.

Second, in any turbulent flow, a small difference in the initial conditions will produce a large difference in the detailed flowfield at a later time. A dramatic example of this was given by Orszag,[13] but, any pair of numerical schemes, when started with the same initial conditions, will produce small differences in the computed flowfields after the first few time steps. These differences will grow, and we cannot expect direct comparison of the results to produce anything meaningful. Thus, we cannot hope to produce an exact numerical simulation of a realization of a turbulent flowfield. Of course, the same effects occur in experiments. Small uncontrolled perturbations can make large differences in the details of a flow.

Third, the entire question of what is to be regarded as a valid simulation is still open. In particular, we need to know how various low-order statistical quantities are affected by changes in other low-order statistical quantities. Are they also affected by more detailed and less obvious changes? These questions lie at the core of our understanding of turbulence and are under intensive investigation. The various workers in turbulence computation have all used methods of their own and in view of the foregoing, clearly defined comparisons are difficult to make.

With these caveats in mind, we point out some of the elements of computational methods that generally are agreed upon. The equations on which the computations are based are clearly conservative of mass, momentum, and kinetic energy, i.e., in the absence of body forces and viscous effects, the only way the integrals of these quantities over a control volume can change is by flow through the boundaries. It is of utmost importance that the spatial finite differences be such as to maintain these properties numerically. Even apparently small deviations from exact conservation produce terrible results in relatively few time steps.

The effect of time differencing on property conservation is more difficult to control but, fortunately, it is typically much smaller than the effect of spatial differencing and can be neglected. Normally, the time step is chosen so as to make the error introduced by the time differencing approximately the same as that caused by the space differencing; one then has a "balanced" computation. The choice of a time method is dictated by the tradeoff between the increased cost per step (in computation and/or storage) of high-accuracy methods and the large time step allowed by such methods. When the choice is made on this basis, stability becomes a yes/no question, i.e., we are interested in knowing only that the method is stable for the time step chosen. There is no reason to choose a method with extra stability if the cost is higher.

Nearly all workers have used second-order explicit methods and the Adams-Bashforth method,

$$u^{n+1} = u^n + \frac{\Delta t}{2}(3u_i^n - u_i^{n-1}) \tag{15}$$

is by far the most popular. In some of our computations, we have found it advantageous to use third- or fourth-order methods, (see Clark et al.[11]). These require more storage, but allow a larger time step for a given number of executed instructions and accuracy. They are used to best advantage in problems in which the computational mesh is so large as to require the use of disk in any case; the penalty incurred is then relatively minimal.

Returning to the question of spatial differencing, the first choice that must be made is the number of grid points (or Fourier modes) to be used. These computations can be done only on very large computers, and the amount of computer time available usually is limited, so the number of points is limited by this resource. (Note that the cost scales at least like N^3.) There is also an advantage in having the number of grid points in at least some of the dimensions a power of two for reasons we will see later. Thus, the numbers of mesh points are typically 16^3 or 32^3, with 64^3 being used in a few cases.

Having chosen the number of mesh points, we next must choose the grid size. Until recently, we believed that the proper approach was to maximize the amount of turbulent energy that is captured by the grid, but we recently have found that there are discrete structures of importance even in the simplest homogeneous flows. These structures, which are not yet completely understood, seem to be responsible for important energy transfers and higher-order statistics. The spatial region that is represented in the computation should be large enough to capture many of these, if the phenomena they are responsible for are to be represented accurately. Generally, this requires a larger grid than the previous criterion.

Both staggered and regular meshes have been used. The former has the advantage of providing somewhat more accuracy, but it has more difficulty with boundary conditions. The latter is also easier to extend to higher order. Deardorff and Schumann have preferred the staggered mesh; we have used both. As mentioned earlier, Orszag uses a Fourier method.

If the calculation were done without filtering, there normally would be a significant amount of energy at the highest wavenumbers. Any numerical method will have a lot of difficulty in handling the high wavenumbers, and the calculation will run into trouble. This difficulty can be looked at in a number of ways. First, no numerical method (other than the Fourier method) will treat these terms accurately. Second, nonlinear interactions among these waves will produce waves with wavenumbers greater than $k_{\max}$, and these will be misinterpreted numerically as small wavenumber waves. The result is a spurious transfer to small wavenumbers (aliasing), unless special precautions are taken. The primary effect of filtering is to reduce the size of the high wavenumber components, thus stabilizing the computation.

From these considerations, it is clear that the width of the filter Δ should be related to the mesh size h. However, they need not be equal – we have found that for a regular mesh $\Delta \simeq 2h$ is required, whereas for the staggered mesh $\Delta \simeq \sqrt{2}h$ is sufficient. Note that the effect of filtering is both to change the initial energy spectrum and to introduce the Leonard term; we found it necessary to include both effects to achieve a successful simulation. As we mentioned earlier, one sometimes can allow the truncation error to represent the Leonard term, and many schemes appear to accomplish this successfully. Essentially, they combine into a single step what we do in several steps. This has advantages, but it also reduces the flexibility somewhat.

A wide range of finite-difference approximations have been tried. Deardorff and Schumann used the standard staggered mesh scheme. We have used a variety of fourth-order schemes. The original rationale for this was that the Leonard and subgrid scale terms are of second order in Δ, but arguments given earlier indicate that this may not be correct. More recently, we have experimented with Fourier space differencing methods which do not require explicit retention of the Leonard term. These methods may be found in the references; space does not permit their inclusion here.

We are now in a position to discuss the manner in which the calculations actually are carried out. An initial field is constructed with the aid of a random number generator. For study of the decay of homogeneous isotropic turbulence (the simplest of all flows) one constructs a divergence free initial field with the correct filtered experimental initial energy spectrum. For other flows, the initial field contains the approximate mean flow plus some isotropic turbulence.

To advance to the next time step, we note that the continuity equation (2) does not contain a time derivative, and special treatment therefore is required. The commonly used technique is explained most easily by writing Eq. (6) as

$$\frac{\partial \bar{u}_i}{\partial t} = F_i(\bar{u}_i) - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{16}$$

where $F(\bar{u}_i)$ represents the convective term (including the Leonard and subgrid scale terms arising from it) and the viscous term. If we then apply a simple first-order (Euler) difference for the time derivative, the difference form is

$$\bar{u}_i^{n+1} = \bar{u}_i^n + \Delta t\left[F_i(\bar{u}_i^n) - \frac{1}{\rho}\frac{\partial \bar{p}^n}{\partial x_i}\right] \tag{17}$$

Now, taking the divergence of this equation, we have

$$\frac{\partial u_i^{n+1}}{\partial x_i} = \frac{\partial u_i^n}{\partial x_i} + \Delta t\left[\frac{\partial F_i^n}{\partial x_i} - \frac{1}{\rho}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\bar{p}^n\right] \tag{18}$$

If we now assume that the flowfield at time step n satisfies continuity, the first term on the right-hand side is zero (if it is nonzero, it can be treated as a small correction term). The object is to make the left-hand side equal zero, so we set it to zero and obtain a Poisson equation for the pressure

$$\nabla^2 \bar{p}^n = \rho \frac{\partial}{\partial x_i} F_i^n \tag{19}$$

The pressure field at step n thus is adjusted so as to cause the velocity field at step $n+1$ to satisfy continuity. One proceeds by advancing the velocity field using the momentum equation and then solving the Poisson equation for the pressure. This holds for other time differencing methods as well.

A little care is necessary here. We earlier stated the importance of having the conservation properties satisfied exactly in the numerical formulation. For this to be the case, it is necessary that the numerical gradient operator used to approximate $\partial \bar{p}/\partial x_i$ be the same operator as that used to define the divergence; otherwise, energy conservation is not achieved. (An exception occurs if a forward difference operator is used for the divergence; a backward difference then must be used for the gradient.) For this reason, one is not free to difference Eq. (19) in an arbitrary manner.

Solution of the Poisson equation (19) can be achieved by a number of methods. Most of the earliest workers used successive over-relaxation (SOR), one of the best iterative methods. In the last few years, however, fast Fourier transform (FFT) methods have become much more popular, especially in problems in which the geometry permits easy application of the FFT. FFT methods are considerably faster,

noniterative, and produce the exact solution to within roundoff error, which makes correction for nonzero divergence unnecessary. They work best when the number of mesh points in a given direction is a power of two and are the reason for this choice of the number of points.

An alternative approach (called the artificial compressibility method) to time advancement was given by Chorin.[14] In this method, one uses the compressible form of the continuity equation and iteratively adjusts the pressure field to eliminate the divergence of the velocity field. Because this method is iterative, it is slower than the method described earlier and has not found wide use.

Finally, mention must be made of the Fourier methods developed by Orszag and co-workers. In this approach, one begins by taking the Fourier transform of Eqs. (1) and (2).

$$\frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k},t) + \sum_{\boldsymbol{k}'} ik_j \hat{u}_i(\boldsymbol{k}')\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}') = \frac{1}{\rho} ik_i \hat{p} - \nu k^2 \hat{u}_i \tag{20a}$$

$$k_i \hat{u}_i = 0 \tag{20b}$$

where a carat denotes the Fourier transform of a variable. Taking the scalar product of Eq. (20a) with u_j and using Eq. (20b), one can solve directly for $\hat{p}$. Substituting the result back into Eq. (20a) one has

$$\frac{\partial \hat{u}_i}{\partial t}(\boldsymbol{k},t) + i \sum_{\boldsymbol{k}'} \left(\delta_{j\ell} - \frac{k_j k_\ell}{k^2}\right) k_i \hat{u}_i(\boldsymbol{k}')\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}') = -\nu k^2 u_i \tag{21}$$

which does not contain the pressure at all. One then solves this equation by a standard explicit method (the viscous term can be treated implicitly). The convolution sum is evaluated by using FFT methods. Special methods for reducing aliasing error also have been developed.

This method yields greater accuracy for N Fourier modes than the grid method with N grid points. For this reason, the method is faster for a given accuracy. It has the disadvantage, however, of being strictly applicable only to simple geometries and has difficulty in treating nonperiodic boundary conditions. For these cases, Orszag suggests that expansions of the variables in orthogonal polynomials may be applicable. If this is so, the method can be extended to other problems, but the relative inflexibility of the method makes it doubtful that this approach will yield a useful general purpose method. Its usefulness is likely to be in cases in which many computations need to be done in a single fixed geometric configuration; the atmospheric problem is thus the leading candidate for large-scale application of the method.

IV. Results

As is clear from some of the remarks in the previous sections, various workers have adopted different attitudes toward large eddy simulations of turbulence and have tackled very different types of problems. To a large degree, these approaches complement each other and cover the range of applications of the method from the purely theoretical to the very applied.

This review will concentrate on our work at Stanford, but a few results from other workers will be given to illustrate the possibilities. The major theme of our work has been to tackle the simplest possible flows first, to get a handle on the validity of the numerical methods and to minimize the number of possible causes of discrepancy. After a simpler flow has been simulated successfully, we will add one additional complicating effect at a time in an attempt to study both the physical and numerical effects of the added factor. Although the approach is slow, it does provide a firm groundwork for the method. We believe that the mixed success of previous authors is because of the necessity of trying to sort out too many effects simultaneously.

The first problem chosen was homogeneous isotropic turbulence, and two approaches were used. In the first, we studied a case at very low Reynolds number using a large number of grid points so that subgrid scale modeling was not needed. Starting with an experimentally determined energy spectrum, the program was asked to predict the decay of the turbulence. The predicted energy decay and spectrum agreed with the experimental data (Corrsin and Comte-Bellot[15]) and the results therefore were accepted as a realization of a turbulent flowfield. We then used these results to test the validity of the subgrid scale modeling. Regarding the computed field as exact, we filtered the results and then could explicitly compute $\bar{u}_i$ and the various filtered quantities that occur in Eqs. (9-13). We also can compute the model prediction for these quantities and compare them directly. Correlation coefficients were used as measures of goodness of fit. We found the subgrid scale Reynolds stresses $\overline{u_i' u_j'}$ correlated at only about 0.3, independent of which model for the eddy viscosity was used (including a turbulence kinetic energy model of the kind used by Schumann). Apparently, the reason is that the principal axes of $\overline{u_i' u_j'}$ are not aligned with the principal axes of the large eddy strain tensor. This also is known to be the case for the Reynolds stress tensor in a shear flow. This suggests that what one learns about subgrid scale modeling applies nearly as well to ensemble-average Reynolds stress modeling. It should be noted, however, that the energy dissipation produced by the model correlated at 0.7 with the exact result. Finally, the results were used to predict the value of the subgrid scale constant by comparing the rms values of the exact and model quantities. Details of this work are given in Clark et al.[11]

The second approach used higher Reynolds numbers and smaller grids (16^3) so that modeling was required. In this case, the model constants were chosen to yield the proper energy decay and the agreement with the values found by the foregoing procedure was very good. It was these calculations which produced the information about proper filter width given in the previous section. The results were again excellent for the spectrum (see Fig. 1) and, of course, for the energy

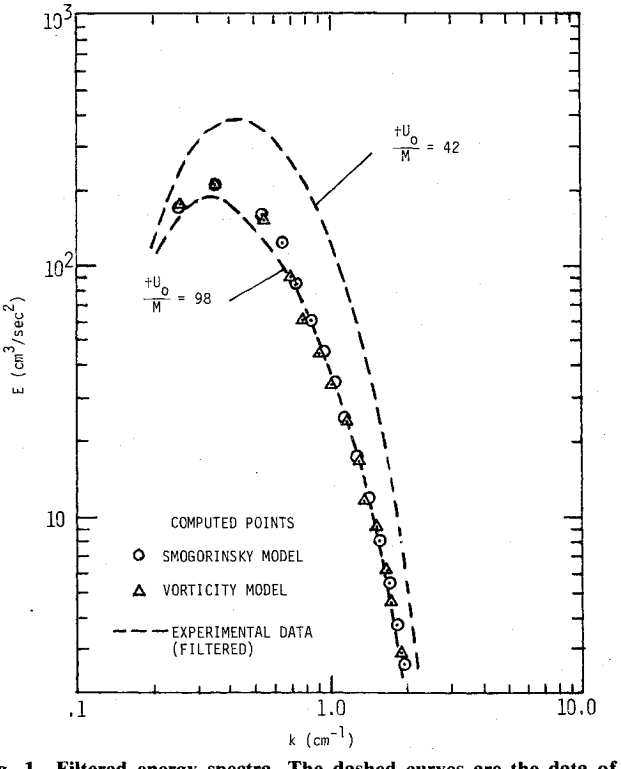

Fig. 1 Filtered energy spectra. The dashed curves are the data of Comte-Bellot and Corrsin[15]; the points are the computational results with two subgrid scale models. In this calculation $\Delta = 2h = 1.5$ cm.

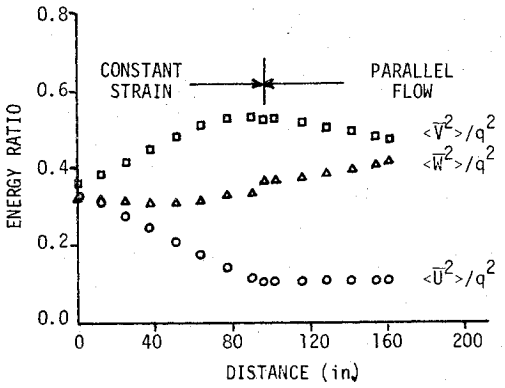

Fig. 2 The development of the velocity components in strained turbulence.

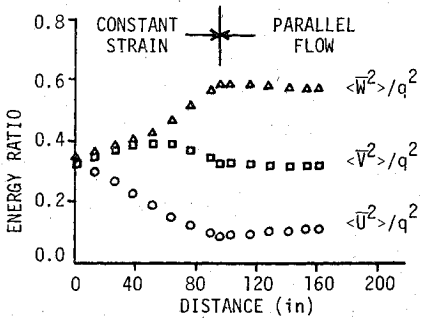

Fig. 3 Same as Fig. 2 but a different set of random numbers was used to generate the initial conditions.

decay. Higher-order statistical quantities, most importantly the skewness, $S_i = \langle(\partial u_i/\partial x_i)^3\rangle/\langle(\partial u_i/\partial x_i)^2\rangle^{3/2}$, also were computed, but filtering makes them difficult to compare with experiment. The skewness values in different directions ($i=1,2,3$) and for initial conditions obtained with different random number sets fluctuate by about 20%. As mentioned earlier, this is apparently caused by the lack of a sufficient number of high skewness structures being captured and can be cured by using a larger grid size. The differences in skewness do not appear to affect the energy spectrum or magnitude, but have other consequences discussed earlier.

Next, the effects of homogeneous strain and shear were added. No changes were made in the method used. The trends obtained were in accord with what is found experimentally and analytically. However, the values of $\langle\tilde{u}_i^2(t)\rangle$ were considerably different in different realizations. We believe that this is because of the effect of pressure redistribution. The equations for $\langle\tilde{u}_i\tilde{u}_j\rangle$, which can be found, for example, in Hinze,[16] contain the pressure-strain term $\langle\tilde{p}(\partial\tilde{u}_i/\partial x_j + \partial\tilde{u}_j/\partial x_i)\rangle$ which tends to transfer energy from the large components of $\langle\tilde{u}_i\tilde{u}_j\rangle$ to the small ones. Since, from Eq. (19) we see that $\tilde{p}$ is quadratic in $\partial\tilde{u}/\partial x$, the pressure strain term is cubic in $\partial\tilde{u}/\partial x$, i.e., it is sensitive to quantities similar to the skewness. Thus, if these quantities are large, energy transfers will tend to reduce the energy of the large components of $\langle\tilde{u}_i\tilde{u}_j\rangle$ (which are also the ones that grow most rapidly) and reduce the net turbulence energy production rate. The results are shown in Figs. 2 and 3. We believe the differences are because of the effects mentioned earlier and can be cured by a change in mesh size. This conjecture will be tested in the near future as part of our program to sort out the relationships among the various parameters. The results can be used to test models of the pressure strain term. In light of the foregoing, the results must be regarded as preliminary, but they seem to show that the models are qualitatively but not quantitatively correct, i.e., they predict trends.

Further calculations were done adding the effect of rotation (Coriolis force). These gave considerable insight into the

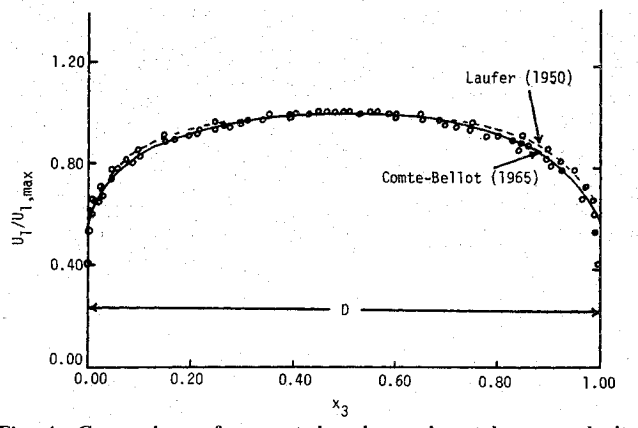

Fig. 4 Comparison of computed and experimental mean velocity profile for turbulent channel flow; results of four different computer runs included. (from Schumann[12]).

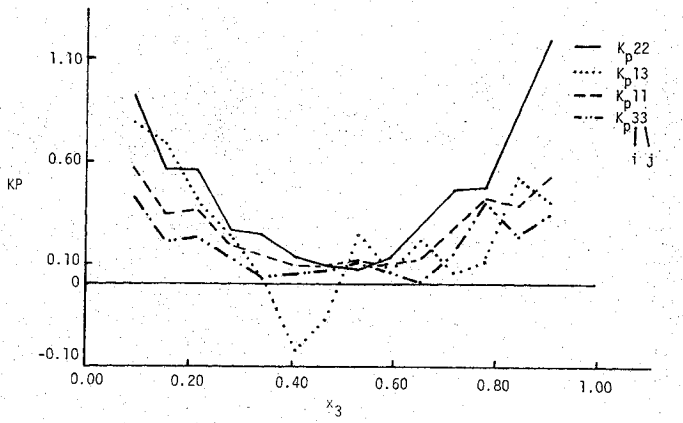

Fig. 5 The "constant" for a model of the pressure strain term. Rotta's model predicts $p(\partial u_i/\partial x_j + \partial u_j/\partial x_i) = -k_p[\langle u_i u_j\rangle - 1/3\langle u_i u_j\rangle\delta_{ij}]$. The prediction of the variation of k_p across the channel is an application of LES methods to model development.

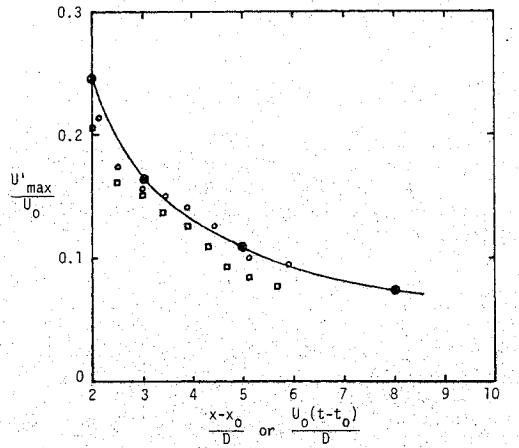

Fig. 6 Axial variation of maximum turbulence intensity in the wake of a self-propelled body. The line is experimental data; the circles and squares are computed by Orszag and Pao.[18]

nature of turbulence production, and showed that the effect of rotation is to inhibit the production of turbulence (see Ferziger and Shaanan[17]). Extensions of this work to include the effects of solid boundaries and other inhomogeneities are now in progress.

Deardorff and Schumann applied the method to turbulent channel flow. For various reasons, the calculation could not be carried to the wall and the boundary therefore was treated by matching the solution to a logarithmic profile at the appropriate distance from the wall. The profiles obtained are in

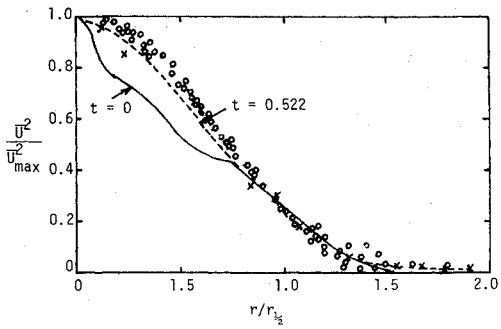

Fig. 7 Radial variation of turbulent intensity in the flow described in Fig. 6. Circles are experimental data; crosses are computed.

excellent agreement with experimental data (see Fig. 4). Schumann also was able to use his results to test models for turbulence quantities and showed that certain "constants" are more appropriately treated as functions of the distance from the wall (see Fig. 5). Deardorff has made a number of meteorological applications of the method to the atmospheric boundary layer with good results, but space does not permit their inclusion here.

As a final example, we mention some of the results obtained by Orszag using the Fourier method. He has studied the decay of isotropic turbulence (with excellent results) and also has treated the wake of a self-propelled body (no net momentum content). The results are shown in Figs. 6 and 7.

All of the calculations of the Stanford group were carried out on CDC-7600 computers. Running times are approximately 2 min for 16^3, 15 min for 32^3, and 90 min for 64^3 (for which some special treatment was used).

V. Concluding Remarks

Large eddy simulation methods are only in the initial stages of development, but they already have displayed a great deal of potential as important tools for understanding and predicting turbulent flows. The results obtained illustrate that these methods can yield valuable results with a minimum of empiricism, something which has been difficult to achieve with methods now in more widespread use.

For the near-term future, large eddy simulations appear to promise the ability to test models used in other types of simulation and to provide constants and other data required to make these computations successful. In this regard, LES methods should result in a reduction in the size of the experimental data base required to benchmark models and thus offer the hope of partially replacing expensive experimental work with less costly computation.

Over the longer range, it is possible that LES methods will become a standard computational tool. Although they are likely to remain expensive, they could find use as final design check methods and as calibrators of day-to-day design tools. The role which they will play in this regard will depend on the extent to which the decrease in effective cost of computation can be maintained over the next decade or two. There is as yet no clear indicator of technological limits being reached, and it may be reasonable to expect cost reductions of an order of magnitude or two in the next decade. If this is so, LES methods should play an increasingly important role in the spectrum of computational techniques.

It is also clear that much remains to be done in the development of LES methods. There is room for a great deal of inventiveness and a great many problems need to be overcome before the methods become effective tools. Thus, we believe that this is a fertile ground for work in the foreseeable future.

Acknowledgments

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